UNIVERSIDAD DE PALERMO

Cálculo Exponencial
Exponential Calculus

Alejandro D. Popovsky, E.M.M.

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Buenos Aires - Argentina
Mario Bravo 1302 - Te.: 963-1179/80 - Fax: 963-1560

EXPONENTIAL CALCULUS

INTRODUCTION

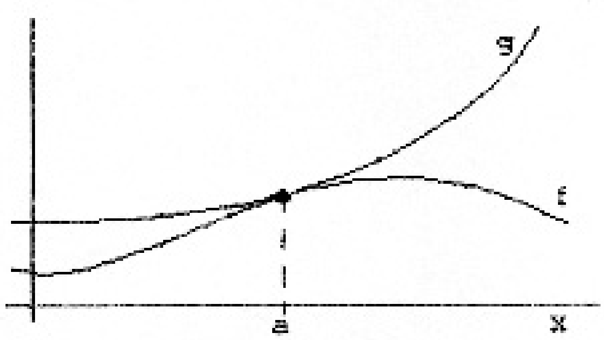
There are two fundamental tools in infinitesimal calculus : differentiation and integration .The first of them is related to the slope concept and the linear functions.The second one is related to the concept of area,but also can be thought of as a continuous sum .

This article studies two new tools and their properties .The first of them is related to the concept of the base of an exponential function .The second one can be thought of as a continuous product .

The relation between the ordinary calculus and the one developed in the following pages ,is analyzed at the end of this article ;besides ,the possibility of developing new tools similar to those studied here, is analyzed .

BASE

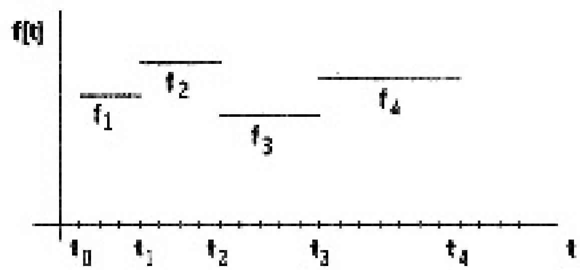
In the same way that we can approximate a function near a point **a**,with a linear function : $f(x) \simeq f(a)+f'(a).(x-a)$,we can approximate it with an exponential function : $f(x) \simeq f(a)\ b^{(x-a)}$.The parameter that we have to adjust for this approximation to be valid ,is the constant **b** .This constant will be called the base of **f** at **a** .(Later on we will define what we understand as a valid approximation.) If **g** were our approximation for the function **f** near the point **a** ,it would look like this :



At the end of this article we will give as an example ,a description of the behavior of two functions by using an exponential approximation .These functions are :the Gauss distribution ,and the gamma function .We will see that for these functions it is more natural ,an exponential description than a linear description .

PRODUCTION

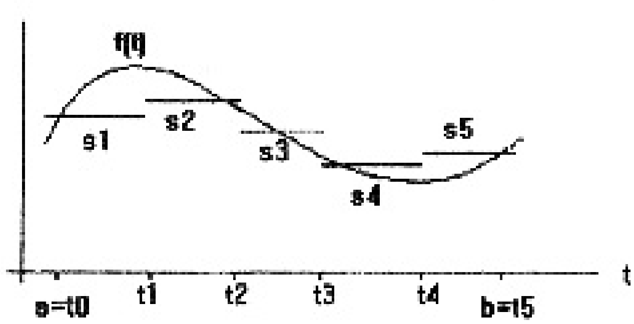
The concept of production may be assimilated to a continuous prouct in the same way that an integral may be asimilated to a continuous sum .But the production can't be associated to something as simple as the area,therefore to introduce this concept we will need an example .
Suppose we have a bacteria population in a lab .We see that in a unit of time the population multiplies itself by a number **f** that we will call reproduction coefficient .
So if we have at the time **t0** a population **p(t0)=p0** ,and the reproduction coefficient remains constant until the time **t1**, then we will have at **t1** a population : $p(t1)=p0.f^{(t2-t0)}$,that is ,the population will grow in an exponential way .
The reproduction coefficient represents the base of the exponential function that describes the growth.
Suppose that the reproduction coefficient depends on some variables controlled at the lab ,then **f** is not constant ,but may depend on time .
If f is a stepped function of time (as in the following graph) ,



then the population at the time **t4** may be calculated with the following product :

$$p(t4)=p0.[f1^{(t1-t0)}.f2^{(t2-t1)}.f3^{(t3-t2)}.f4^{(t4-t3)}].$$

But if **f** varies in a continuous way ,we will have to approximate **f** with a stepped function **s** ,that takes the values **s1,s2,...** at the different steps ,



,and then obtain an approximated value for the population **at** the time **t=b** ,by calculating a product similar to the one we have just seen :

$$p(b) \simeq p0.[s1^{(t1-t0)}.s2^{(t2-t1)}...s5^{(t5-t4)}]$$

This is only an approximated value ,but meanwhile we reduce the length of the steps of our approximation for **f** ,the product approaches more and more to the exact value .

The limit that the product tends to , when the lenght of the steps tends to zero ,will be named :the production of **f** along **[a,b]** .(We will give a more rigorous definition later on.)

Definition 1

A function $f:R \to R$ is exponentiable at a point b, if $f(b) \neq 0$, and there exist a real number $\lambda>0$ and a real function μ, continuous at b, such that :

$$1. \quad \frac{f(x)}{f(b)} = \lambda^{x-b} \cdot \mu(x) \qquad\qquad 2. \quad \lim_{x \to b} \mu(x)^{1/(x-b)} = 1 \quad .$$

The function $E(x) = \lambda^{(x-b)}$, will be called exponential of f at b. We will use the notation : $\lambda = f^{\sim}(b)$, and call this number : base of f at b.
The second condition assures us that $\mu(x)$ tends to 1 at b, faster than $E(x)$, because :

$$\lim_{x \to b} E(x)^{1/(x-b)} = \lambda \quad .$$

This implies that when x is close to b, the following approximation is valid:

$$\frac{f(x)}{f(b)} \approx E(x) \qquad ,\text{or :} \qquad f(x) \approx f(b) \cdot E(x) \quad .$$

We can calculate $f^{\sim}(b)$ with the following limit :

$$f^{\sim}(b) = \lim_{x \to b} \left[\frac{f(x)}{f(b)}\right]^{1/(x-b)} \quad .$$

We can see from the condition 1, that f is equal to the multiplication of continuous functions at b : $f(x) = f(b) \cdot E(x) \cdot \mu(x)$, i.e. if a function is exponentiable at a point b, it is continuous there too.
The following theorem gives us an alternative definition of exponentiability.

Theorem 1

The function f is exponentiable at a, if and only if there exists the limit :

$$\lim_{x \to a} \left[\frac{f(x)}{f(a)}\right]^{1/(x-a)} \qquad ,\text{and it is finite and nonzero .}$$

Demonstration :

1) If f is exponentiable at a, then there exist a number $\lambda>0$ and a continuous function μ that meet the conditions 1 and 2 of the definition 1, then :

$$\lim_{x \to a} \left[\frac{f(x)}{f(a)}\right]^{1/(x-a)} = \lim_{x \to a} \lambda \cdot \mu(x)^{1/(x-a)} = \lambda \quad .$$

2) If the enunciated limit exists and it is finite and nonzero, then f must be continuous and nonzero at a, and the limit must be equal to some positive number λ. Let's define the following function :

$$\mu(x) = \frac{f(x)}{f(a)} \cdot \lambda^{-(x-a)} \quad .$$

It is easy to see that this function is continuous at a and it meets the first condition of the definition 1. So also it meets the second condition, because :

$$\lim_{x \to a} \mu(x)^{1/(x-a)} = \lim_{x \to a} \left[\frac{f(x)}{f(a)}\right]^{1/(x-a)} \cdot \lambda^{-1} = \lambda \cdot \lambda^{-1} = 1 \quad .$$

The next theorems will show us two important properties of the base .

<u>Theorem 2</u>

Let f be exponentiable at a, then $g(x) = \alpha \cdot f(x)^{\beta}$ is exponentiable at a , and :

$$\bar{g}(a) = [\bar{f}(a)]^{\beta}$$

$(\alpha \neq 0)$

Demonstration

f is exponentiable at a , then $f(a) \neq 0$, and conditions 1 and 2 of the definition 1 are satisfied by a number $\lambda > 0$ and a continuous function μ, then :

$$\frac{g(x)}{g(a)} = \frac{\alpha \cdot f(x)^{\beta}}{\alpha \cdot f(a)^{\beta}} = \lambda^{\beta \cdot (x-a)} \cdot \mu(x)^{\beta} = \delta^{x-a} \cdot \sigma(x)$$

$(\delta = \lambda^{\beta}$ and $\sigma(x) = \mu(x)^{\beta})$
i.e. the function g meets the condition 1. So also , it meets condition 2 , because :

$$\lim_{x \to a} \sigma(x)^{1/(x-a)} = \lim_{x \to a} \mu(x)^{\beta/(x-a)} = 1$$

And we have : $\bar{g}(x) = \sigma = \lambda^{\beta} = [\bar{f}(x)]^{\beta}$.

<u>Theorem 3</u>

If f and g are both exponentiable at a, then $(f \cdot g)$ is exponentiable at a , and :

$$\overline{(f \cdot g)}(a) = \bar{f}(a) \cdot \bar{g}(a)$$

Demonstration

The hypothesis tells us there exist $\lambda 1$ and $\mu 1$ that meet the conditions 1 and 2 of the definition 1 for the function f . And there exist $\lambda 2$ and $\mu 2$ that meet the same conditions for the function g. Then :

$$\frac{(f \cdot g)(x)}{(f \cdot g)(a)} = \lambda 1^{x-a} \cdot \lambda 2^{x-a} \cdot \mu 1(x) \cdot \mu 2(x) = \lambda^{x-a} \cdot \mu(x) \quad .$$

$(\lambda = \lambda 1 \cdot \lambda 2$, $\mu(x) = \mu 1(x) \cdot \mu 2(x)$). And we can see that :

$$\lim_{x \to a} \mu(x)^{1/(x-a)} = \lim_{x \to a} \mu 1(x)^{1/(x-a)} \cdot \mu 2(x)^{1/(x-a)} = 1 \quad .$$

i.e. $(f \cdot g)$ is exponentiable at a and :

$$\overline{(f \cdot g)}(a) = \lambda = \lambda 1 \cdot \lambda 2 = \bar{f}(a) \cdot \bar{g}(a) \quad .$$

Theorems 2 and 3 show us that the base is a linear transformation based on the operations product and exponentiation .That is :

$$\left[f(x)^{\alpha}\, g(x)^{\beta}\right]^{-} = f^{-}(x)^{\alpha}\, g^{-}(x)^{\beta}$$

This is better explained at the end of this article .We will see now the behavior of the base of a function at the extreme points .

Theorem 4

Let f be a function defined in some neighborhood of c,it has an extreme point at c,and is exponentiable at c ,then $f^{-}(c) = 1$.

Demonstration

f is exponentiable at c ,then $f(c) \neq 0$ and f is continuous at c.Then there exists a neighborhood of c, for which f doesn't change sign.
f has an extreme point at c,this means f has an absolute extreme point at some neighborhood of c.
We choose a neighborhood of c: A(c) for which these two requirements are met.
We will analyze four different cases separately :
Case 1) f has a maximum at c and is positive in A(c).
Given h such that $c+h \in A(c)$,then $f(c) \geq f(c + h)$,this means :

$$\frac{f(c+h)}{f(c)} \leq 1 \quad .$$

If $h>0$: $\left[\frac{f(c+h)}{f(c)}\right]^{1/h} \leq 1$. Then $\lim_{h\to 0+} \left[\frac{f(c+h)}{f(c)}\right]^{1/h} \leq 1$

If $h<0$: $\left[\frac{f(c+h)}{f(c)}\right]^{1/h} \geq 1$. Then $\lim_{h\to 0-} \left[\frac{f(c+h)}{f(c)}\right]^{1/h} \geq 1$.

Both limits must be the same ,because f is exponentiable at c,hence $f^{-}(c)=1$.

Case 2) f has a minimum at c and is positive in A(c).
Given h such that $c+h \in A(c)$,then $f(c) \leq f(c + h)$,this means :

$$\lim_{h\to 0+} \left[\frac{f(c+h)}{f(c)}\right]^{1/h} \geq 1 \quad \text{,and also} \quad \lim_{h\to 0-} \left[\frac{f(c+h)}{f(c)}\right]^{1/h} \leq 1$$

Again ,both limits must be the same ,hence $f^{-}(c)=1$.

Cases 3 and 4) f has a maximum or a minimum at c ,and is negative in A(c) .
Let $g(x)=(-1)\cdot f(x)$ .Then g has a maximum or a minimum at c,is positive in A(c) ,is exponentiable at c (theorem 2) ,and $g^{-}(c)=f^{-}(c)$ .And we have just seen in cases 1) and 2) that $g^{-}(c)=1$.

Theorem 5

Let f be continuous in [a,b],exponentiable in (a,b) ,and $f(a) = f(b) \neq 0$.
Then there exists a point $c \in (a,b)$ such that $f^{-}(c) = 1$.

Demonstration

Because of f's continuity in [a,b] ,it reaches maximum and minimum in that interval .If the maximum is reached at $c \in (a,b)$ then $f^{\sim}(c) = 1$,because f is exponentiable there .If the minimum is reached at $c \in (a,b)$,again $f^{\sim}(c) = 1$. If the maximum and the minimum are reached at the points a and b ,then it is evident that f is constant in [a,b] ,this means that for every point $x \in (a,b)$ we have : $f^{\sim}(c) = 1$.(It is very easy to show that the base of a constant equals 1)

The next theorem is very similar to the medium value theorem of the differential calculus .

Theorem 6

Let f be continuous in [a,b] ,exponentiable at (a,b) ,and nonzero at a and b. Then there exists a point $c \in (a,b)$ such that :

$$f^{\sim}(c) = \left[\frac{f(b)}{f(a)}\right]^{1/(b-a)} .$$

Demonstration

Let $h(x) = f(x) \cdot \left[\frac{f(a)}{f(b)}\right]^{\frac{x-a}{b-a}}$

h equals the product of two exponentiable functions in (a,b) (because the function which multiplies f is an exponential function),then theorem 3 assures us that h is exponentiable in that interval .Also,h is continuous in [a,b] , and $h(a) = h(b) = f(a)$ .Therefore theorem 5 tells there exists a point $c \in (a,b)$ such that $h^{\sim}(c) = 1$ .On the other side the base of h at c is :

$$h^{\sim}(c) = f^{\sim}(c) \cdot \left[\frac{f(a)}{f(b)}\right]^{1/(b-a)} .$$

And this means :

$$f^{\sim}(c) = \left[\frac{f(b)}{f(a)}\right]^{1/(b-a)} .$$

At this point, you should have probably noticed that our approach to the subject is very similar to that of an elementary calculus course .So also ,the demonstrations should be very familiar (*).This suggests there exists a close relation between these tools and the calculus we already know.
We will analyze this later on.

Theorem 7

Let g be differentiable at a ,and let f be exponentiable at g(a) ,then (fog) is exponentiable at a ,and :

$$(fog)^{\sim}(a) = (f^{\sim}og)(a)^{g'(a)}$$

Demonstration

g is differentible at a ,this means :

$$\cdot\ g(x)-g(a) = \delta\cdot(x-a) + \sigma(x)$$

$$\cdot\ \lim_{x\to a} \frac{\sigma(x)}{x-a} = 0$$

Also, because of f being exponentiable at g(a) we have :

$$\cdot\ \frac{f(y)}{f(g(a))} = \lambda^{y-g(a)}\cdot\mu(y)$$

$$\cdot\ \lim_{y\to g(a)} \mu(y)^{1/(y-g(a))} = 1$$

$(\ \delta = g'(a)\ ;\ \lambda = f^{-}(g(a))\ )$

And because of g differentiability at a we can replace y by g(x) :

$$\frac{f(g(x))}{f(g(a))} = \lambda^{g(x)-g(a)}\cdot\mu(g(x)) = \lambda^{\delta\cdot(x-a)+\sigma(x)}\cdot\mu(g(x))$$

$$= \underbrace{\left[\lambda^{\delta}\right]}_{\alpha}{}^{x-a}\cdot\underbrace{\lambda^{\sigma(x)}\cdot\mu(g(x))}_{\phi(x)} = \alpha^{x-a}\cdot\phi(x)$$

Therefore (fog) meets the first condition of the definition 1 ,and

$$\lim_{x\to a} \phi(x)^{1/(x-a)} = \lim_{x\to a} \lambda^{\sigma(x)/(x-a)}\ \mu(g(x))^{1/(x-a)}$$

$$= \lim_{x\to a} \lambda^{\overbrace{\left[\frac{\sigma(x)}{x-a}\right]}^{\to 0}}\cdot\lim_{x\to a} \overbrace{\left[\mu(g(x))^{\left[\frac{1}{g(x)-g(a)}\right]}\right]}^{\to 1}{}^{\overbrace{\left[\frac{g(x)-g(a)}{x-a}\right]}^{\to g'(a)}} = 1\cdot 1^{g'(a)} = 1\ .$$

Hence the second condition is met too ,(fog) is exponentiable at a .
Consequently

$$(f\circ g)^{-}(a) = \alpha = \lambda^{\delta} = f^{-}(g(a))^{g'(a)} = (f^{-}\circ g)(a)^{g'(a)}$$

Theorem 8

If f is exponentiable at a and g es differentiable en a ,then f^{g} is exponentiable at a ,and :

$$\left[f^{g}\right]^{-}(a) = \left[f^{g'}\right](a)\cdot\left[(f^{-})^{g}\right](a)$$

Demonstration :

$$\cdot\ \frac{\left[f^{g}\right](x)}{\left[f^{g}\right](a)} = \frac{f(x)^{g(a)}}{f(a)^{g(a)}}\cdot f(x)^{g(x)-g(a)}$$

f is exponentiable at a, therefore :

$$\frac{f(x)}{f(a)} = \delta^{x-a} \cdot \mu(x) \quad ; \quad \lim_{x \to a} \mu(x)^{1/(x-a)} = 1 \quad ; \quad \delta = f^{\sim}(a)$$

so also $f(x)^{g(a)}$ is exponentiable at a, and : $\left[f^{g(a)}\right]^{\sim}(a) = f^{\sim}(a)^{g(a)}$

$$\frac{f(x)^{g(a)}}{f(a)^{g(a)}} = \delta^{g(a)(x-a)} \cdot \mu(x)^{g(a)} \quad ; \quad \lim_{x \to a} \mu(x)^{g(a)/(x-a)} = 1$$

. g is differentiable at a, therefore :

$$g(x) - g(a) = \beta \cdot (x-a) + \epsilon(x) \quad ; \quad \lim_{x \to a} \frac{\epsilon(x)}{x-a} = 0 \quad ; \quad \beta = g'(a)$$

. then :

$$\frac{\left[f^{g}\right](x)}{\left[f^{g}\right](a)} = \left[\delta^{g(a) \cdot (x-a)} \cdot \mu(x)^{g(a)}\right] \cdot \left[f(a) \cdot \delta^{x-a} \cdot \mu(x)\right]^{\beta \cdot (x-a)} \cdot f(x)^{\epsilon(x)}$$

$$= \left[\delta^{g(a)} \cdot f(a)^{\beta}\right]^{(x-a)} \cdot \underbrace{\left[\delta^{\beta}\right]^{(x-a)^2} \cdot f(x)^{\epsilon(x)} \cdot \mu(x)^{\beta \cdot (x-a)} \cdot \mu(x)^{g(a)}}_{\pi(x)}$$

$$= \lambda^{x-a} \cdot \pi(x)$$

$$\lim_{x \to a} \pi(x)^{1/(x-a)} = \lim_{x \to a} \underbrace{\left[\delta^{\beta}\right]^{(x-a)}}_{\to 1} \cdot \underbrace{f(x)^{\epsilon(x)/(x-a)}}_{\substack{\to 1 \\ [*]}} \cdot \underbrace{\mu(x)^{\beta}}_{\to 1} \cdot \underbrace{\mu(x)^{g(a)/(x-a)}}_{\to 1} = 1$$

[*] f is continuous and positive at a, and $\epsilon(x)/(x-a) \to 0$.

Therefore f^{g} is exponentiable at a, and :

$$\left[f^{g}\right]^{\sim}(a) = \lambda = \left[\delta^{g(a)} \cdot f(a)^{\beta}\right] = f^{\sim}(a)^{g(a)} \cdot f(a)^{g'(a)}$$

The following are the bases of some common functions,the demonstration is very simple if a theorem that we will see later is applied .

$$\left[f(x)^{\alpha}\cdot g(x)^{\beta}\right]^{*} = g^{*}(x)^{\alpha}\cdot f^{*}(x)^{\beta}$$

$$(\alpha)^{*} = 1$$

$$(x)^{*} = e^{1/x}$$

$$(x+\alpha)^{*} = e^{1/(x+\alpha)}$$

$$\left[\alpha^{x}\right]^{*} = \alpha$$

$$\left[\alpha^{g(x)}\right]^{*} = \alpha^{g'(x)}$$

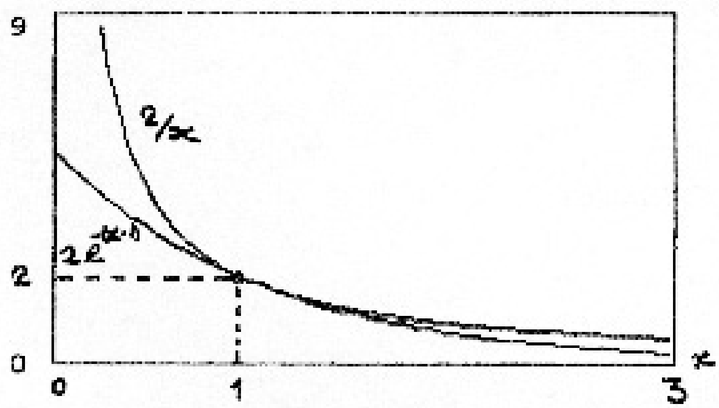
$$\left[x^{x}\right]^{*} = e\cdot x$$

$$(\sin(x))^{*} = e^{\mathrm{cotg}(x)}$$

Now, we will see an exponential approximation for $f(x) = 2/x$,close to the point $x0=1$:

$$f^{*}(x)= e^{-1/x} \quad ; \; f^{*}(x0)=e^{-1} \quad ; \; f(x0)=2$$

Therefore : $f(x) \approx 2\cdot e^{-(x-1)}$



Sometimes it is easier to find the extreme points of a function using the base than using the derivative,for example ,we will find the extreme points of the function :

$$f(x) = x^{n}\cdot e^{-x} \quad \text{in the interval } (0,\infty)\ .$$

The base of f in this interval is:

$$f^{*}(x) = e^{n/x}\cdot e^{-1} = e^{(n/x-1)}$$

Hence $f^{*}(x) = 1$ at $x = n$,this means $x=n$ is an extreme point for f .
We will see another intresting example after seeing production and some other properties of the base .

Production

Let f be a definite, bounded and positive (d.b.p.) function in [a,b]. Let I be a partition of [a,b]. The letter S will denote the generic subinterval of I, and L(S) will denote the length of S.
If I and I' are partitions of [a,b], we will say I' is finer than I, if each subinterval S of I equals the union of several subintervals S_k of I'.

We will need the following definitions :

$$m_S(f) = \min\{ f(x) \ / \ x \in S \} \qquad M_S(f) = \max\{ f(x) \ / \ x \in S \}$$

$$L(f,I) = \prod_S m_S(f)^{L(S)} \qquad U(f,I) = \prod_S M_S(f)^{L(S)}$$

(We always choose the positive value of the exponentiation .)

Theorem 1 : $L(f,I) \le U(f,I)$

We omit the demonstration of this theorem because it is obvious .

Theorem 2 :

If I' is finer than I, then : $L(f,I) \le L(f,I')$; $U(f,I') \le U(f,I)$

Demonstration : each subinterval S of I equals the union of some S_k of I' .

Therefore : $L(S) = \sum_k L(S_k)$; so also : $m_S(f) \le m_{S_k}(f)$

Therefore :
$$m_S(f)^{L(S)} = m_S(f)^{\sum_k L(S_k)} = \prod_k m_S(f)^{L(S_k)} \le \prod_k m_{S_k}(f)^{L(S_k)}$$

This is valid for every S of I, it follows from this : $L(f,I) \le L(f,I')$
The second inequality is demonstrated in an analogous way .

Theorem 3

For any pair of partitions of [a,b] : I and I', we have :

$$L(f,I) \le U(f,I')$$

Demonstration : Let I" be finer than I and finer than I', then :

$$L(f,I) \le L(f,I'') \le U(f,I'') \le U(f,I') \quad .$$

We need to define the following sets :

$$R(f,a,b) = \{ L(f,I) \ / \ I \text{ is a partition of } [a,b] \}$$

$$T(f,a,b) = \{ U(f,I) \ / \ I \text{ is a partition of } [a,b] \}$$

Theorem 3 assures us that $\sup R(f,a,b) \leq \inf T(f,a,b)$
(supA denotes the lowest upper bound for the set A; and infA denotes the highest lower bound .)

Definition

Let f be d.b.p. in [a,b]. Then f is producible in [a,b] if and only if : supR(f,a,b) = infT(f,a,b) . In this case we will denote this value with :

$$\mathcal{P}_{[a,b]} f$$

We can also write : $\mathcal{P}_{a}^{b} f$ o $\mathcal{P}_{a}^{b} f(x)^{dx}$.

The following theorem is an alternative definition for producibility, that will be very useful in several demonstrations.

Theorem 4

Let f be d.b.p in [a,b] , then f is producible in [a,b] if and only if , for any $\varepsilon>0$ we can find a partition I of [a,b] such that :

$$\frac{U(f,I)}{L(f,I)} < e^{\epsilon}$$

Demonstration :

1) If for any $\varepsilon>0$ there exists a partition I of [a,b] that meets last inequality, then considering that :

$$L(f,I) \le \mathrm{supR}(f,a,b) \le \mathrm{infT}(f,a,b) \le U(f,I)$$

we can assure that :

$$1 \le \frac{\mathrm{infT}(f,a,b)}{\mathrm{supR}(f,a,b)} \le \frac{U(f,I)}{L(f,I)} < e^{\epsilon}$$

Therefore supR(f,a,b) = infT(f,a,b) , i.e. f is producible in [a,b] .

2) If f is producible in [a,b] then supR(f,a,b) = infT(f,a,b) , therefore there exist two partitions of [a,b] : I , I' such that :

$$\frac{U(f,I)}{L(f,I')} < e^{\epsilon}$$

If I" is at the same time finer than I and I' then :

$$L(f,I') \le L(f,I") \le U(f,I") \le U(f,I)$$

Therefore :

$$1 \le \frac{U(f,I")}{L(f,I")} \le \frac{U(f,I)}{L(f,I')} < e^{\epsilon}$$

Now we will demonstrate several important properties of the production of a function .

Theorem 5

If f and g are producible functions in [a,b],then (f·g) is producible in [a,b] ,and :

$$\prod_a^b (f\cdot g) = \left(\prod_a^b f\right)\cdot\left(\prod_a^b g\right)$$

Demonstration :Let I be a partition of [a,b] with general subinterval S ,then for any $x \in S$ it verifies that : $m_s(f)\cdot g(x) \le f(x)\cdot g(x)$,therefore :

$$m_s(m_s(f)\cdot g) = m_s(f)\cdot m_s(g) \le m_s(f\cdot g) \quad \text{, hence :}$$

$$L(f,I)\cdot L(g,I) \le L((f\cdot g),I) \qquad [1]$$

in an analogous way it can be shown that :

$$U((f\cdot g),I) \le U(f,I)\cdot U(g,I) \qquad [2]$$

If f and g are producible in [a,b] ,then for any $\epsilon>0$ there exist two partitions : I , I' such that :

$$\frac{U(f,I)}{l(f,I)} < e^{\epsilon/2} \qquad\qquad \frac{U(g,I')}{l(g,I')} < e^{\epsilon/2}$$

If I" is at the same time finer than I and I' ,then :

$$\frac{U(f,I")\cdot U(g,I")}{L(f,I")\cdot L(g,I")} \le \frac{U(f,I)}{L(f,I)}\cdot\frac{U(g,I')}{L(g,I')} < e^{\epsilon}$$

Therefore (considering [1] and [2]) :

$$\frac{U((f\cdot g),I")}{L((f\cdot g),I")} \le \frac{U(f,I")\cdot U(g,I")}{L(f,I")\cdot L(f,I")} \le e^{\epsilon}$$

Hence (f·g) is producible in [a,b] .

And for any I we have :

$$L(f,I)\cdot L(g,I) \le L((f\cdot g),I) \le \prod_b^a (f\cdot g) \le U((f\cdot g),I) \le U(f,I)\cdot U(g,I)$$

$$L(f,I)\cdot L(g,I) \le \left(\prod_b^a f\right)\cdot\left(\prod_b^a g\right) \le U(f,I)\cdot U(g,I)$$

$\dfrac{U(f,I)}{L(f,I)}$ and $\dfrac{U(g,I)}{L(g,I)}$ can be as close to one as we wish,because f and g are producible functions, then :

$$\prod_a^b (f\cdot g) = \left(\prod_a^b f\right)\cdot\left(\prod_a^b g\right) \quad .$$

Theorem 6

Let $a < b < c$,then f is producible in [a,c] if and only if f is producible in [a,b] and in [b,c] ,in such case :

$$\prod_a^c f = \left(\prod_a^b f\right)\cdot\left(\prod_b^c f\right)$$

Demonstration : Let I be a partition of [a,b] ,and let I' be a partition of [b,c].I" will be a partition of [a,c] consisting of all the subintervals from I and I'.Then :

$$L(f,I'') = L(f,I)\cdot L(f,I') \le U(f,I)\cdot U(f,I') = U(f,I'')$$

1)If f is producible in [a,c] then for any $\epsilon>0$ there exists a partition of [a,c] :I" ,such that :

$$\frac{U(f,I'')}{L(f,I'')} = \left[\frac{U(f,I)}{L(f,I)}\right]\cdot\left[\frac{U(f,I')}{L(f,I')}\right] < e^{\epsilon}$$

Both brackets are greater than one,then both of the must be smaller than e^{ϵ} , Hence f is producible in [a,b] and in [b,c] .

2)If f is producible in [a,b] and in [b,c] then for any $\epsilon>0$ there exist a partition of [a,b] and a partition of [b,c] (I and I' respectively) such that:

$$\frac{U(f,I)}{L(f,I)} < e^{\epsilon/2} \qquad \frac{U(f,I')}{L(f,I')} < e^{\epsilon/2}$$

Therefore :

$$\frac{U(f,I'')}{L(f,I'')} = \left[\frac{U(f,I)}{L(f,I)}\right]\cdot\left[\frac{U(f,I')}{L(f,I')}\right] < e^{\epsilon}$$

This means f is producible in [a,c] .

3) $$L(f,I'') = L(f,I)\cdot L(f,I') \le \left(\int_a^b f\right)\cdot\left(\int_b^c f\right) \le U(f,I)\cdot U(f,I') = U(f,I'')$$

$$L(f,I'') \le \int_a^c f \le U(f,I'')$$

$\dfrac{U(f,I'')}{L(f,I'')}$ can be as close to one as we wish,because f is producible in [a,c].

Therefore :

$$\int_a^c f = \left(\int_a^b f\right)\cdot\left(\int_b^c f\right).$$

Notes:.If we define : $\int_b^a f = 1/\left(\int_a^b f\right)$,then theorem 6 will be valid even though the following inequality is not met : $a < b < c$.

.It follows at once from the theorem 6 ,that if f is producible in [a,b] ,then it is producible in [a,x] (x is any point from [a,b]) .

We are now to calculate the production of a constant $c > 0$:
Let I be a partition of [a,b] with generic subinterval S.
Then $m_S(c) = M_S(c) = c$ .It follows :

$$L(c,I) = \prod_S m_S(c)^{L(S)} = \prod_S c^{L(S)} = c^{\sum_S L(S)} = c^{(b-a)}$$

In an analogous way : $U(c,I) = c^{(b-a)}$, and this is for any I of [a,b] ,then c is producible in [a,b] ,and :
$$\int_a^b c = c^{(b-a)}$$
This result was easy to guess,because it agrees with the definition for production given in the introduction .

<u>Theorem 7</u>

Let f be d.b.p. in [a,b] ,then f is producible in [a,b] if and only if (1/f) is producible there too .

Demonstration :Let I be a partition of [a,b] with generic subinterval S .

Considering that f is d.b.p. in [a,b] : $m_S(f) = \dfrac{1}{M_S(1/f)}$; $M_S(f) = \dfrac{1}{m_S(1/f)}$

Then : $L(f,I) = \dfrac{1}{U(1/f,I)}$; $U(f,I) = \dfrac{1}{L(1/f,I)}$

If f is producible in [a,b] then for any $\epsilon > 0$ there exists a partition of [a,b]: I,such that : $\dfrac{U(f,I)}{L(f,I)} < e^{\epsilon}$

Therefore : $\dfrac{U(1/f,I)}{L(1/f,I)} = \dfrac{1/L(f,I)}{1/U(f,I)} = \dfrac{U(f,I)}{L(f,I)} < e^{\epsilon}$

Hence (1/f) is producible in [a,b].

To demonstrate the reciprocal relation ,we only have to define g=(1/f) .

<u>Theorem 8</u>

If f is producible in [a,b] ,then $\left[f^{\alpha}\right]$ is producible in [a,b] ,and :
$$\int_a^b \left[f^{\alpha}\right] = \left(\int_a^b f\right)^{\alpha}$$
Demonstration : Let g be producible in [a,b] ,and let $\beta > 0$ .Let I be a partition of [a,b].g is positive in [a,b] (because it is producible there) , then :
$$m_S\left[g^{\beta}\right] = m_S(g)^{\beta} \quad \text{,then} \quad L\left[g^{\beta},I\right] = L(g,I)^{\beta} \quad .$$
In an analogous way : $U\left[g^{\beta},I\right] = U(g,I)^{\beta}$.

For any $\epsilon > 0$ there exists a partition of [a,b],such that: $\dfrac{U(g,I)}{L(g,I)} < e^{\epsilon/\beta}$

Therefore : $\dfrac{U\left[g^{\beta},I\right]}{L\left[g^{\beta},I\right]} = \left[\dfrac{U(g,I)}{L(g,I)}\right]^{\beta} < e^{\epsilon}$ .Hence $\left[g^{\beta}\right]$ is producible in [a,b] .

And for any I of [a,b] ,the following is met :

$$L\left[g^{\beta},I\right] = L(g,I)^{\beta} \leq \left(\int_a^b g\right)^{\beta} \leq U(g,I)^{\beta} = U\left[g^{\beta},I\right]$$

$$L\left[g^{\beta},I\right] \leq \int_a^b \left[g^{\beta}\right] \leq U\left[g^{\beta},I\right]$$

$\dfrac{U\left[g^{\beta},I\right]}{L\left[g^{\beta},I\right]}$ can as close to one as we want ,hence :

$$\int_a^b \left[g^{\beta}\right] = \left(\int_a^b g\right)^{\beta}$$

1) $\alpha = 0$.

$$\int_a^b \left[f^{\alpha}\right] = \int_a^b 1 = 1^{(b-a)} = 1$$

$$\left(\int_a^b f\right)^{\alpha} = 1$$

(This is valid because the production is defined for d.b.p. functions ,then for any producible function f we have : $L(f,I) > 0$,this means the production will be positive too.)

2) $\alpha > 0$. In this case we only have to take $g = f$ and $\beta = \alpha$.

3) $\alpha < 0$.

$$\left(\int_a^b f\right)\cdot\left(\int_a^b (1/f)\right) = \int_a^b [f\cdot(1/f)] = \int_a^b 1 = 1$$

$$\Rightarrow \int_a^b \left[f^{-1}\right] = \left(\int_a^b f\right)^{-1}$$

If I take $g = \left[f^{-1}\right]$ and $\beta = -\alpha$ then :

$$\int_a^b \left[f^{\alpha}\right] = \int_a^b \left[\left[f^{-1}\right]^{-\alpha}\right] = \int_a^b \left[g^{\beta}\right] = \left(\int_a^b g\right)^{\beta} = \left(\int_a^b \left[f^{-1}\right]\right)^{-\alpha} = \left(\left(\int_a^b f\right)^{-1}\right)^{-\alpha}$$

$$= \left(\int_a^b f\right)^{\alpha} .$$

Theorem 9

Let f be producible in [a,b] ,and let $f(x) \geq 1$ for any $x \in [a,b]$,then :

$$\int_a^b f \geq 1$$

Demonstration :If I is a partition of [a,b] then for any δ included in I : $1 \leq m_{\delta}(f)$,therefore :

$$1 \leq L(f,I) \leq \sup R(f,a,b) \leq \int_a^b f .$$

Corollary 1

Let f,g be producible in [a,b] ,and let $f \geq g$ for every $x \in [a,b]$ .Then :

$$\int_a^b f \geq \int_a^b g$$

Demonstration : theorem 7 tells us that (1/g) is producible,then theorem 5 assures us that $[f\cdot(1/g)] = (f/g)$ is producible .
$f \geq g$ therefore $(f/g) \geq 1$,then by using the last theorem :

$$\int_a^b g \leq \left(\int_a^b g\right)\cdot\left(\int_a^b (f/g)\right) = \int_a^b [g\cdot(f/g)] = \int_a^b f \quad .$$

Corollary 2

Let f be producible in W = [a,b],then :

$$m_W(f)^{b-a} \leq \int_a^b f \leq M_W(f)^{b-a}$$

Demonstration : it follows at once from corollary 1 .

Theorem 10

Let f be producible in [a,b] .Then $F(x) = \int_a^x f$ is continuous in [a,b].

Demonstration : Theorem 6 and last corollary assure us that F is d.b.p. for any $x \in [a,b]$,
Let $c \in [a,b]$,then :

$$\lim_{x\to c} F(x) = F(c)\cdot\lim_{x\to c}\frac{F(x)}{F(c)} = F(c)\cdot\lim_{x\to c}\frac{\int_a^x f}{\int_a^c f} = F(c)\cdot\lim_{x\to c}\int_c^x f$$

(If c = a or c = b we should take the limit with x approaching c from the inside of the interval .)

If $x > c$: $\quad m_{[a,b]}(f)^{x-c} \leq m_{[c,x]}(f)^{x-c} \leq \int_c^x f \leq M_{[c,x]}(f)^{x-c} \leq M_{[a,b]}(f)^{x-c}$

If $x < c$: $\quad M_{[a,b]}(f)^{x-c} \leq M_{[x,c]}(f)^{x-c} \leq \int_c^x f \leq m_{[x,c]}(f)^{x-c} \leq m_{[a,b]}(f)^{x-c}$

Therefore $\int_c^x f$ can be as close to one as we wish, when x is aproaching c .

Hence : $\lim_{x\to c}\int_c^x f = 1$;and $\lim_{x\to c} F(x) = F(c)$.

Theorem 11

Let f be d.b.p. in [a,b] and continuous in the same interval except for a null content set of points .Then f is producible in [a,b] .

Demonstration :Let c be a point from (a,b) such that f is continuous at c . Then for any $\epsilon > 0$ there exists a neighborhood V of c ,such that for any point $x \in V$,it is verified that :

$$f(c)\cdot e^{-\epsilon/2} < f(x) < f(c)\cdot e^{\epsilon/2}$$

Then :
$$1 \le \frac{M_V(f)}{m_V(f)} < \frac{f(c)\cdot e^{\epsilon/2}}{f(c)\cdot e^{-\epsilon/2}} = e^{\epsilon}$$

Let B be the set of points for which f is discontinuous,then B has null content,this means that there exists a finite collection of closed intervals : $U_1, U_2, \ldots, U_k$ which covers B,such that $\sum_i L(U_i) < \epsilon$.

Let I be a partition of $[a,b]$ such that each S of I belongs to one of the following sets :

Φ_1 : intervals S of I such that $S \subseteq U_i$ for some i .

Φ_2 : intervals S of I such that $S \cap B$ is null.

Then :
$$\prod_{S \in \Phi_1} \left[\frac{M_S(f)}{m_S(f)}\right]^{L(S)} \le \prod_{S \in \Phi_1} \left[\frac{M_{[a,b]}(f)}{m_{[a,b]}(f)}\right]^{L(S)} = \left[\frac{M_{[a,b]}(f)}{m_{[a,b]}(f)}\right]^{\sum_{S\in\Phi_1} L(S)}$$

$$\le \left[\frac{M_{[a,b]}(f)}{m_{[a,b]}(f)}\right]^{\sum_i L(U_i)} < \left[\frac{M_{[a,b]}(f)}{m_{[a,b]}(f)}\right]^{\epsilon}$$

If $S \in \Phi_2$ then for any $x \in S$ there exists an interval V_x which contiains x , and :

$$\frac{M_{V_x}(f)}{m_{V_x}(f)} < e^{\epsilon}$$

S is compact then there exist a finite number of intervals $V_{x_1}, \ldots, V_{x_m}$ that covers S .Let I'be a partition of S such that each S' of I' is included in some V_{x_i} ,then :

$$\frac{M_{S'}(f)}{m_{S'}(f)} < e^{\epsilon}$$

Therefore :

$$\frac{U(f,I')}{L(f,I')} = \prod_{S'} \left[\frac{M_{S'}(f)}{m_{S'}(f)}\right]^{L(S')} < \prod_{S'} e^{\epsilon\cdot L(S')} = e^{\epsilon\cdot L(S)}$$

Hence there exists a partition I" of [a,b] with subintervals S" ,finer than I, such that if $S \in \Phi_2$:

$$\prod_{S''\subseteq S} \left[\frac{M_{S''}(f)}{m_{S''}(f)}\right]^{L(S'')} < e^{\epsilon \cdot L(S)}$$

Therefore :

$$\frac{U(f,I'')}{L(f,I'')} = \left[\prod_{S''\subset S\in\Phi_1} \left[\frac{M_{S''}(f)}{m_{S''}(f)}\right]^{L(S'')}\right]\left[\prod_{S''\subset S\in\Phi_2} \left[\frac{M_{S''}(f)}{m_{S''}(f)}\right]^{L(S'')}\right]$$

$$\leq \left[\frac{M_{[a,b]}(f)}{m_{[a,b]}(f)}\right]^{\epsilon} \cdot \prod_{S\in\Phi_2} e^{\epsilon\, L(S)} \leq \left[\frac{M_{[a,b]}(f)}{m_{[a,b]}(f)}\right]^{\epsilon} \cdot e^{\epsilon\,(b-a)}$$

Then $\dfrac{U(f,I'')}{L(f,I'')}$ can be as close to one as we wish,consequently f is producible in [a,b] .

Theorem 12

Let f be continuous and positive in [a,b] then there exists a point $c \in [a,b]$ such that :

$$\int_a^b f = f(c)^{(b-a)}$$

Demonstration : Last theorem tells us that f is producible in [a,b] ,and theorem 9 tells us that in such case :

$$m_{[a,b]}(f)^{b-a} \leq \int_a^b f \leq M_{[a,b]}(f)^{b-a}$$

Then there exists a positive number α such that $m_{[a,b]}(f) \leq \alpha \leq M_{[a,b]}(f)$,and :

$$\alpha^{b-a} = \int_a^b f$$

f is continuous in [a,b] ,consequently for any value α between $m_{[a,b]}(f)$ and $M_{[a,b]}(f)$ there exists a point $c \in [a,b]$ such that $f(c) = \alpha$.

The following theorem could be named the fundamental theorem of the exponential calculus,and it will show us the close relation between the production and the base .

Theorem 13

Let f be continuous and positive in [a,b] then the function $F(x) = \int_a^x f$ is exponentiable in (a,b) and $F^* = f$

Demonstration : f is continuous and positive in [a,b] ,then it is producible that interval ,and so it is in [a,x] ,for every $x \in [a,b]$;then F is definite for every point in [a,b].If c and x are in [a,b] then :

$$\frac{F(x)}{F(c)} = \frac{\int_a^x f}{\int_a^c f} = \int_c^x f = f(z)^{x-c} = \lambda^{x-c} \mu(x)$$

$z \in [c,x]$, $\lambda = f(c)$,and $\mu(x) = \left[\frac{f(z)}{f(c)}\right]^{x-c}$

Now :
$$\lim_{x\to c} \mu(x)^{1/(x-c)} = \lim_{x\to c} \frac{f(z)}{f(c)} = \frac{1}{f(c)} \lim_{z\to c} f(z) = \frac{f(c)}{f(c)} = 1$$

(If c = a or c = b we should take the limit from the inside of the interval .)
Hence F is exponentiable and : $F^* = f$.

Theorem 14

Let f be producible in [a,b] ,and let $f = g^*$ for some function g in [a,b] . Then :

$$\int_a^b f = \frac{g(b)}{g(a)}$$

Demonstration :Let I be a partition of [a,b] with generic subintervals : $[t_{i-1},t_i]$.Theorem 6 from the base properties tells that for each subinterval i, there exists a point x_i belonging to that interval such that :

$$\frac{g(t_i)}{g(t_{i-1})} = g^*(x_i)^{t_i - t_{i-1}} = f(x_i)^{t_i - t_{i-1}}$$

Consequently :

$$L(f,I) = \prod_i m_{[t_{i-1},t_i]}(f)^{t_i - t_{i-1}} \le \prod_i f(x_i)^{t_i - t_{i-1}} = \prod_i \frac{g(t_i)}{g(t_{i-1})} = \frac{g(b)}{g(a)}$$

In an analogous way : $\frac{g(b)}{g(a)} \le U(f,I)$ .Therefore : $L(f,I) \le \frac{g(b)}{g(a)} \le U(f,I)$
for any I of [a,b] .Then considering that f is producible in [a,b] :

$$\int_a^b f = \frac{g(b)}{g(a)}$$

The following two theorems will be very useful for calculating productions .
Any function F that satisfies $F^{*} = f$ will be called (exponential) primitive of f .
Theorem 13 assures us that any function f continuous in an interval [a,b] , has a primitive in that interval ,for example :

$$F(x) = \prod_{a}^{x} f$$

<u>Theorem 15</u>

If f and g' are continuous then : $\displaystyle\prod_{g(a)}^{g(b)} f = \prod_{a}^{b} (f_{\circ}g)^{g'}$

Demonstration : If F is a primitive of f then :

$$\prod_{g(a)}^{g(b)} f = F(x)\Big]_{g(a)}^{g(b)} = \frac{F(g(b))}{F(g(a))} = \frac{(F_{\circ}g)(b)}{(F_{\circ}g)(a)}$$

The function $(F_{\circ}g)$ is a primitive of : $(F^{*}_{\circ}g)^{g'} = (f_{\circ}g)^{g'}$.

Consequently : $\displaystyle\prod_{a}^{b} (f_{\circ}g)^{g'} = (f_{\circ}g)\Big]_{a}^{b} = \frac{(f_{\circ}g)(b)}{(f_{\circ}g)(a)}$

<u>Theorem 16</u>

If f^{*} and g' are continuous then :

$$\prod_{a}^{b} (f^{*})^{g} = \left(f^{g}\Big]_{a}^{b}\right) \Big/ \left(\prod_{a}^{b} f^{g'}\right) \quad \text{,and :} \quad \prod_{a}^{b} f^{g'} = \left(f^{g}\Big]_{a}^{b}\right) \Big/ \left(\prod_{a}^{b} (f^{*})^{g}\right)$$

Demontration :If f is exponentiable and g is differentiable then :

$$\left[f^{g}\right]^{*} = (f^{*})^{g} \cdot f^{g'}$$

Therefore: $\displaystyle\prod_{a}^{b} \left[f^{g}\right]^{*} = \prod_{a}^{b} \left[f^{g'} \cdot (f')^{g}\right] = \left(\prod_{a}^{b} f^{g'}\right) \cdot \left(\prod_{a}^{b} (f^{*})^{g}\right)$

$$\prod_{a}^{b} \left[f^{g}\right]^{*} = f^{g}\Big]_{a}^{b} \quad .$$

We are now to see how to calculate the base of a function if we already know its derivative,and how to calculate its production knowing its integral .

<u>Teorema 17</u>

f is exponentiable at a if and only if ln|f| is differentiable at a.In such case :

$$\ln(f^{*}) = (\ln|f|)'$$

Demonstration :
1) If f is exponentiable at a then $f(a) \neq 0$,and there exist a positive number λ and a function $\mu(x)$ continuous at a ,such that :

$$\left[\frac{f(x)}{f(a)}\right] = \lambda^{x-a} \cdot \mu(x) \quad , \quad \lim_{x \to a} \mu(x)^{1/(x-a)} = 1$$

f is continuous at a (because it is exponentiable there), then there exists a neighborhood of a for which f does not change sign, then :

$$\ln\left[\frac{f(x)}{f(a)}\right] = \ln\left|\frac{f(x)}{f(a)}\right| = \ln|f(x)| - \ln|f(a)|$$

$$\ln\left[\frac{f(x)}{f(a)}\right] = (x-a)\cdot\ln\lambda + \ln\mu(x) = (x-a)\cdot\delta + \pi(x)$$

($\delta = \ln\lambda$,y $\pi(x) = \ln\mu(x)$)

And : $$\lim_{x\to a}\frac{\pi(x)}{x-a} = \lim_{x\to a}\ln\left[\mu(x)^{1/(x-a)}\right] = \ln 1 = 0$$

Therefore $\ln|f|$ is differentiable at a and : $(\ln|f|)'(a) = \delta = \ln\lambda = \ln(f^{*}(a))$

2) $\ln|f|$ is differentiable at a then ther exist a number δ and a function $\pi(x)$ continuous at a such that :

$$\ln|f(x)| - \ln|f(a)| = (x-a)\,\delta + \pi(x) \quad , \qquad \lim_{x\to a}\frac{\pi(x)}{x-a} = 0$$

Because of $\ln|f|$ differentiability at a, f is continuous and nonzero at a . Then there exists a neighborhood of a for which f does not change sign , in that neighborhood :

$$\ln|f(x)| - \ln|f(a)| = \ln\left|\frac{f(x)}{f(a)}\right| = \ln\left[\frac{f(x)}{f(a)}\right]$$

$$\ln|f(x)| - \ln|f(a)| = \ln\left[e^{\delta\cdot(x-a)}\cdot e^{\pi(x)}\right] = \ln\left[\lambda^{x-a}\cdot\mu(x)\right]$$

($\lambda = e^{\delta}$,y $\mu(x) = e^{\pi(x)}$)

Therefore : $$\frac{f(x)}{f(a)} = \lambda^{x-a}\cdot\mu(x)$$

$$\lim_{x\to a}\mu(x)^{1/(x-a)} = \lim_{x\to a} e^{\pi(x)/(x-a)} = e^{0} = 1$$

Consequently f is exponentiable at a ,and :

$$\ln(f^{*}(a)) = \ln\lambda = \ln\left[e^{\delta}\right] = \delta = (\ln|f|)'(a) \quad .$$

Theorem 18

f is producible en [a,b] if and only if lnf is integrable in that interval, in such case :

$$\ln \mathcal{P}_a^b f = \int_a^b \ln f$$

(We write lnf in the place of $\ln|f|$ because we have only defined production for positive functions .)

Demonstration :We will need some definitions .Let I be a partition of [a,b] with generic subinterval S, then :

$$H(f,I) = \sum_{S} m_s(f)\cdot L(S) \qquad\qquad K(f,I) = \sum_{S} M_s(f)\cdot L(S)$$

The logarithm function has always a positive slope,then if f is positive in some interval V : $m_v(\ln f) = \ln m_v(f)$; $M_v(\ln f) = \ln M_v(f)$.
Therefore :

$$H(\ln f,I) = \sum_{S} m_s(\ln f)\cdot L(S) = \sum_{S} \ln m_s(f)\cdot L(S) = \sum_{S} \ln\left[m_s(f)^{L(S)}\right]$$

$$= \ln \prod_{S} m_s(f)^{L(s)} = \ln L(f,I)$$

In an analogous way we can show that : $K(\ln f,I) = \ln U(f,I)$.

1) If f is producible in [a,b] then for any $\epsilon > 0$ there exists a partition I of [a,b] such that :

$$\frac{U(f,I)}{L(f,I)} < e^{\epsilon}$$

Therefore : $K(\ln f,I) - H(\ln f,I) = \ln U(f,I) - \ln L(f,I) = \ln\left[\frac{U(f,I)}{L(f,I)}\right] < \epsilon$.

Consequently lnf is integrable in [a,b] .

2) If lnf is integrable in [a,b] then for any $\epsilon > 0$ there exists a partition I of [a,b] such that :

$$K(\ln f,I) - H(\ln f,I) < \epsilon$$

Then : $$\frac{U(f,I)}{L(f,I)} = e^{\ln U(f,I)-\ln L(f,I)} = e^{K(\ln f,I)-H(\ln f,I)} < e^{\epsilon}$$

Consequently f is producible in [a,b] .

3)If f is producible in [a,b] then :

$$\ln L(f,I) = H(\ln f,I) < \int_a^b \ln f < K(\ln f,I) < \ln U(f,I)$$

$$\ln L(f,I) < \ln \mathcal{P}_a^b f < \ln U(f,I)$$

L(f,I) can be as close to U(f,I) as we wish ,then :

$$\ln \mathcal{P}_a^b f = \int_a^b \ln f \quad .$$

We are now going to see as an example, the production of some simple functions:

$$\int_a^b c^{dx} = c^{b-a}$$

$$\int_a^b \left[f(x)^{\alpha} \cdot g(x)^{\beta}\right]^{dx} = \left(\int_a^b f^{dx}\right)^{\alpha} \cdot \left(\int_a^b g^{dx}\right)^{\beta}$$

$$\int_a^b x^{dx} = \left[\left[\frac{x}{e}\right]^x\right]_a^b = b^b \cdot a^{-a} \cdot e^{-(b-a)}$$

$$\int_a^b (x+\alpha)^{dx} = \left[\left[\frac{x}{e}\right]^x\right]_{a+\alpha}^{b+\alpha} = (b+\alpha)^{b+\alpha} \cdot (a+\alpha)^{-(a+\alpha)} \cdot e^{-(b-a)}$$

$$\int_a^b \left[x^2 - 1\right]^{dx} = \left(\int_a^b (x+1)^{dx}\right) \cdot \left(\int_a^b (x-1)^{dx}\right) = \left(\left[\left[\frac{x}{e}\right]^x\right]_{a+1}^{b+1}\right) \cdot \left(\left[\left[\frac{x}{e}\right]^x\right]_{a-1}^{b-1}\right)$$

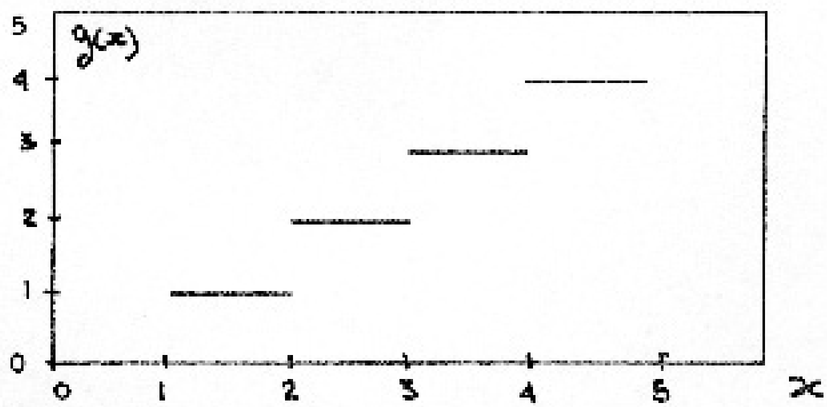
$$= \left[\frac{x+1}{x-1} \cdot \left[\frac{x^2-1}{e^2}\right]^x\right]_a^b = \frac{b+1}{b-1} \cdot \frac{a-1}{a+1} \cdot \frac{\left[b^2-1\right]^b}{\left[a^2-1\right]^a} \cdot e^{-2 \cdot (b-a)}$$

Later we will see the production of a polynomial with complex zeros .
But now we will study the behavior of the base of an important function .

When x is a natural number ,the gamma function can be calculated in the following way :

$$\Gamma(x) = (x-1)! = \prod_{n=1}^{x-1} n = \int_1^x g(x)^{dx}$$

g(x) is a stepped function,that for each x ,assumes the value of the higher natural number, equal or lower than x .By now we may presume that the base of Γ is a function similar to g(x).



By using the Stirling forumla ,we can approximate (for large values of x) :

$$\Gamma(x+1) \approx \sqrt{2\cdot\pi\cdot x}\cdot(x/e)^{x}$$

Therefore :
$$(\Gamma(x+1))^{\sim} \approx e^{1/(2x)}\cdot x$$

On the other side :
$$(\Gamma(x+1))^{\sim} = (x\cdot\Gamma(x))^{\sim} = e^{1/x}\cdot\Gamma^{\sim}(x)$$

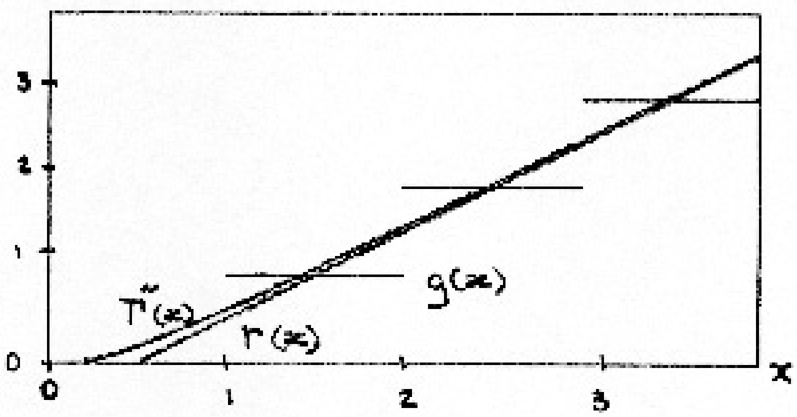
Consequently :
$$\Gamma^{\sim}(x) \approx e^{-1/(2x)}\cdot x$$

The asymptote of this function is $r(x) = x-1/2$,because

$$\lim_{x\to\infty}\left[e^{-1/(2x)}\,x - r(x)\right] = 0$$

$$\lim_{x\to\infty}\left[\left[e^{-1/(2x)}\,x\right]' - r'(x)\right] = 0$$

And as we have supposed, $r(x)$ is the best approximation for $g(x)$.



Suppose that f is an exponentiable function,and so is the base of f.Then the base of the base of f is going to be called :the second order base of f.In an analogous way we can define the nth order base of f .
Suppose that f is three times exponentiable at a.Then we can define the following function :

$$g(x) = f(a)\cdot f^{\sim}(a)^{x-a}\cdot f^{\approx}(a)^{\left[\frac{(x-a)^2}{2!}\right]}\cdot f^{\stackrel{\approx}{\sim}}(a)^{\left[\frac{(x-a)^3}{3!}\right]}$$

(where $f^{\approx}(x)$ is the base of $f^{\sim}(x)$,and $f^{\stackrel{\approx}{\sim}}(x)$ is the base of $f^{\approx}(x)$)

It easy to show that the first,second and third order bases of g at a equals the bases of f .For this reason we may suppose that the function g is a good approximation for f near the point a.If we want a better approximation for f,we could add more factors to our product .This is not alway possible because f could not be exponentiable all the times we wish .

Now we are going to give as an example ,an exponential approximation for the function f(x) = cos(x) ,near the point x = 0 .Considering that :

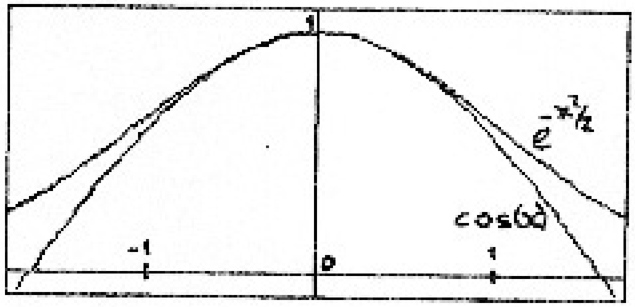
$$f^{\sim}(x) = \exp(-\tan(x)) \qquad f^{\approx}(x) = \exp(-1 - \tan^2(x))$$

(These bases are very easy to demonstrate if theorem 17 is applied)

Then : $f(0) = 1$; $f^{\sim}(0) = 1$; $f^{\approx}(0) = e^{-1}$

Therefore :

$$f(x) \approx f(0)\cdot f^{\sim}(0)^{x}\cdot f^{\approx}(0)^{\left[\frac{x^2}{2}\right]} = e^{-\left[\frac{x^2}{2}\right]}$$



It is intresting to observe that if we integrate cos(x) and our exponential approximation,we obtain the following :

$$\int_0^x \exp\left[\frac{-x^2}{2}\right] dx \approx \sin(x)$$

The maximum relative error between this integral and sin(x) in the interval [-1,1] is 2% ,this let us appreciate the quality of our exponential approximation .

These exponential approximations are very similar to the Taylor sums,for this reason,they could be called "Taylor products " .

ABOUT THE CALCULUS STRUCTURE

Up to now you should probably have noticed that the base and the derivative have many similarities ,and the same happens with the production and the integral .Here we will try to generalize these concepts ,to achieve deeper knowledge about them .

Let A be the set of all the real functions with real domain .Then we can construct an Abelian group ,joining this set with the addition operation :(A,+) ,as a proof ,we see that : the sum of two of these functions ,is another function
from A;the sum of these functions is associative and commutative ,there exists a neutral function (the null function);and for every function f belonging to A there exists a simetrical function f^{s} that belongs to A too.($f^{s} = -f$).
This group (A,+) structures itself in a vector space over the body of the real numbers K, through the (external composition law) product : (A,+,K,*) , as a proof ,we see that :
If f belongs to A ,and α belongs to K then $f*\alpha$ belongs to A; the following distributive laws are valid : $f*(\alpha+\beta) = f*\alpha+f*\beta$;$(f+g)*\alpha = f*\alpha+g*\alpha$ (α,β belong to K; f,g belong to A),the mixed associative law is valid : $f*(\alpha*\beta) = (f*\alpha)*\beta$;and the neutral element for the scalar product remains neutral for the external product : $f*1 = f$ (1 belongs to K and f belongs to A) .

Let B be the set of all the positive functions with real domain .Then we can construct an Abelian group ,joining this set with the product operation:(A,*) ,as a proof ,we see that : the product of two of these functions ,is
another function from B;the product of these functions is associative and commutative ,there exists a neutral function (the unity function);and for every function f belonging to B there exists a simetrical function f^{p} that belongs to B too.($f^{p} = 1/f$).
This group (B,*) structures itself in a vector space over the body of the real numbers K, through the (external composition law) exponentiation :(B,*,K,^),
as a proof ,we see that :
If f belogs to B,and α belongs to K then $f\hat{}\alpha$ belongs to B ;the following distribuive laws are valid: $f*(\alpha+\beta) = f*\alpha+f*\beta$;$(f*g)\hat{}\alpha = f\hat{}\alpha*g\hat{}\alpha$ (α,β belong to k ; f,g belong to B);the mixed associative law is valid : $f\hat{}(\alpha*\beta) = (f\hat{}\alpha)\hat{}\beta$;and the neutral element for the scalar product remains neutral for the external exponentiation : $f\hat{}1 = f$ (1 belongs to K and f belongs to B) .

Let's see the derivative definition for functions belonging to (A,+,K,*) :

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} (f(x+h) + f^{s}(x)) * (1/h)$$

(f,f',f^{s} belong to A ;and x,h belong to K)

Now ,let's see the base definition for function belonging to (B,*,K,^) :

$$f^{\sim}(x) = \lim_{h\to 0} \left[\frac{f(x+h)}{f(x)}\right]^{1/h} = \lim_{h\to 0} (f(x+h) * f^{p}(x)) \hat{}\ (1/h)$$

($f,f^{\sim},f^{p}$ belong to B ;and x,h belong to K)

Now it is easy to see that the base and the derivative are the same transformation ,but performed in different spaces .The same ocurrs between the production and the integral .
In other words we have moved the calculus structure from one space to another.
That is why the base and the derivative properties are so similar .
It is easy to see that there exists an isomorphism between the two spaces,and this one is the exponential function,because : exp(α*f+β*g) =exp(f)^α*exp(g)^β .The last theorem ,the one that relates the base with the derivative and the production with the interval was based on this .
Nothing stops us from thinking that the calculus structure can't be moved to other spaces ,with different sets of functions ,and different composition laws.

For example ,in a vector space of functions : (C,&,K,@),Where (C,&) is an abelian group for which ,for any function f belonging to C there exists a simmetrical function $f^{\&}$,related to the operation & . We can associate for every function f in C ,another function :

$$f^{\circ}(x) = \lim_{h\to 0} (f(x+h)\ \&\ f^{\&}(x))\ @\ (1/h)$$

($f, f^{\circ}, f^{\&}$ belong to C; and x,h belong to K)
It is evident that some of the requirements needed for the validity of the definition above,are : that the functions belonging to C have their domains included in the body K ,and that the limit definition makes sense in this space.

As an example we are going to move the calculus to the space: (D,*,C,^) ,
D is the set of all the functions with complex domain ,applied in the all the complex plane ,except for the point zero ,and C is the complex body .* is the complex product ,and ^ is some univalued determination for the exponentiation
Now we will calculate the production of a polynomial with complex zeros that was pending :

$$\int_0^1 \left[x^2+1\right]^{dx} = \int_0^1 ((x+i)\cdot(x-i))^{dx} = \left(\int_0^1 (x+i)^{dx}\right)\cdot\left(\int_0^1 (x-i)^{dx}\right)$$

$$= \left(\int_i^{1+i} x^{dx}\right)\cdot\left(\int_{-i}^{1-i} x^{dx}\right) = \left(\left[\left[\frac{x}{e}\right]^x\right]_i^{1+i}\right)\cdot\left(\left[\left[\frac{x}{e}\right]^x\right]_{-i}^{1-i}\right)$$

$$= \left[\left[\frac{x+i}{e}\right]^{(x+i)}\cdot\left[\frac{x-i}{e}\right]^{(x-i)}\right]_0^1$$

$$= \left[e^{-2\cdot(x+\operatorname{arccotg}(x))}\cdot\left[x^2+1\right]^x\right]_0^1$$

$$= 1.302$$

Even though there were complex partial results ,it was presumable that the final result was going to be real,because the production we wanted to calculate was a real one .